Solvable rational extensions of the isotonic oscillator

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Combining recent results on rational solutions of the Riccati-Schrödinger equations for shape invariant potentials to the finite difference Bäcklund algorithm and specific symmetries of the isotonic potential, we show that it is possible to generate the three infinite sets ($L1$, $L2$ and $L3$ families) of regular rational solvable extensions of this potential in a very direct and transparent way.

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I. INTRODUCTION

In quantum mechanics, the rarity of the potentials which are exactly solvable in closed-form (most of them belonging to the class of shape-invariant potentials[1–3]) gives a undeniable importance to the reseach of new families of such potentials. A possible way to generate new solvable potentials is to start from the known ones and to construct regular rational extensions of them. If the procedure has a long history, in the last years important progress have been made in this direction[4–18]. In a recent work[19] we proposed an approach allowing to generate such regular extensions starting from every translationally shape-invariant potential (TSIP) of the second category (as defined in[20]). For this, we use regularized excited states Riccati-Schrödinger (RS) functions as superpotentials in a generalized SUSY partnership. The regularization scheme corresponds to a "spatial Wick rotation" which eliminates the singularities from the real axis, a device already suggested by Shnol'[21] in 1994 as a way to generate rational extensions of the harmonic potential. In the following years, this suggestion has been developped by Samsonov and Ovcharov[22] and Tkachuk[23]. Recently Fellows and Smith[24] rediscovered this technique in the case of the harmonic oscillator, the second rational extension of which being the so-called CPRS potential[25]. In[19], we have extended the procedure to cover the whole set of TSIP belonging to the second category. For the isotonic oscillator, we recovered the $L1$ family of rational extensions discovered by Gomez-Ullate, Kamran and Milson[4–8], Quesne[9–12] and Odake, Sasaki et al[13,16,17]. For the other second category potentials, the infinite set of regular quasi-rational extensions that we obtain coincides with the $J1$ family[4–7,9–13,16,17].

In the present article, combining the finite difference Bäcklund algorithm with new regularization schemes which are based on specific symmetries of the isotonic potential, we show how the extension of the SUSY QM partnership to excited states allows to generate the three infinite sets $L1$, $L2$ and $L3$ of regular rationally solvable extensions of the isotonic potential (as well as the singular $L0$ and $L3$ ones) in a direct and systematic way. This approach leads to a simple and transparent proof of the shape-invariance of the potentials of the $L1$ and $L2$ series.

The paper is organized as follows. We first recall how the generalization of the SUSY partnership based on excited states leads to a series of singular rational extensions of the initial potential. We then introduce basic elements concerning the finite difference Bäcklund algorithm viewed as a set of covariance transformations for the class of Riccati-Schrödinger equations and we interpret the generalized SUSY partnership in this perspective. In the third and fourth sections, we recapitulate some results concerning the isotonic oscillator, its connection with confluent hypergeometric equation and the Kienast-Lawton-Hahn's Theorem which describes the distribution of the zeros of the Laguerre functions on the real axis. The fifth section is devoted to present the set of parameters transforms which are discrete symmetries of the isotonic potential. Using them as regularization transformations, we show then that the finite difference Bäcklund algorithm based on the corresponding regularized RS functions generates directly the three series $L1$, $L2$ and $L3$ of regular rationally solvable extensions of the isotonic potential. In the last section, we prove the shape-invariance of the potentials of the $L1$ and $L2$ series.

II. GENERALIZED SUSY PARTNERSHIP BASED ON EXCITED STATES: $L0$ SERIES OF RATIONAL EXTENSIONS

Consider a family of closed form exactly solvable hamiltonians $H(a) = -d^2/dx^2 + V(x;a),\ a \in \mathbb{R}^m,\ x \in I \subset \mathbb{R}$, the associated bound states spectrum of which being given by $(E_n(a), w_n(x;a))$, where $w_n(x;a) = -\psi_n'(x;a)/\psi_n(x;a)$

is the Riccati-Schrödinger (RS) function associated to the n^{th} bound state eigenfunction $\psi_n(x;a)$. The Riccati-Schrödinger (RS) equation[20] for the level $E_n(a)$ is then

$$-w_n'(x;a)+w_n^2(x;a)=V(x;a)-E_n(a), \tag{1}$$

where we suppose $E_0(a)=0$. The RS function presents n real singularities associated to the n simple nodes of the eigenstates $\psi_n(x;a)$. As it is well known[26,27], $H(a)$ admits infinitely many different factorizations of the form

$$H(a)-E_n(a)=A^+\left(w_n\right)A\left(w_n\right), \tag{2}$$

where

$$A\left(w_n\right)=d/dx+w_n(x;a), \tag{3}$$

with, in particular

$$A\left(w_n\right)\psi_n(x;a)=0. \tag{4}$$

This allows to associate to $H(a)$ or $V(x;a)$ an infinite family of partners given by

$$H^{(n)}(a)-E_n(a)=A\left(w_n\right)A^+\left(w_n\right)=-d^2/dx^2+V^{(n)}(x;a), \tag{5}$$

with

$$V^{(n)}(x;a)=V(x;a)+2w_n'(x;a). \tag{6}$$

For $n\geq 1$, these potentials are all singular at the nodes of $\psi_n(x;a)$ and are defined on open intervals only. On these domains, $H^{(n)}(a)$ is (quasi)isospectral to $H(a)$. Indeed, writing

$$\psi_k^{(n)}(x;a)=A\left(w_n\right)\psi_k(x;a), \tag{7}$$

it is easy to verify that we have for any k

$$H^{(n)}(a)\psi_k^{(n)}(x;a)=E_k(a)\psi_k^{(n)}(x;a), \tag{8}$$

that is, $\psi_k^{(n)}(x;a)$ is an eigenstate of $H^{(n)}(a)$ associated to the eigenvalue $E_k(a)$. We write symbolically

$$V^{(n)}(x;a)\underset{iso}{\equiv}V(x;a), \tag{9}$$

where $\underset{iso}{\equiv}$ means "isospectral to". Defining

$$w_{n,k}(x;a)=-\frac{\psi_k^{(n)\prime}(x;a)}{\psi_k^{(n)}(x;a)}, \tag{10}$$

Eq.(8) gives, for $k>n$

$$-w_{n,k}'(x;a)+w_{n,k}(x;a)^2=V^{(n)}(x;a)-E_k(a). \tag{11}$$

This scheme generalizes the SUSY QM partnership, by using the excited state RS functions w_n as superpotentials. However, only for the ground state $n=0$, the factorization and then the partner potential $V^{(0)}(a)=V(x;a)+2w_0'(x;a)$ are non singular and we recover the usual SUSY QM partnership[1,2].

III. FINITE DIFFERENCE BÄCKLUND ALGORITHM

We can consider the preceding partnership in a different way which gives a prominent role to the covariance transform of the RS equations class.

A. Invariance group of the Riccati equations

As established by Cariñena et al.[28,29], the finite-difference Bäcklund algorithm is a consequence of the invariance of the set of Riccati equations under a subset of the group $\mathcal{G}$ of smooth $SL(2,\mathbb{R})$-valued curves $Map(\mathbb{R}, SL(2,\mathbb{R}))$. For any element $A \in \mathcal{G}$ characterized by the matrix:

$$A(x) = \begin{pmatrix} \alpha(x) & \beta(x) \\ \gamma(x) & \delta(x) \end{pmatrix}, \quad \det A(x) = \alpha(x)\delta(x) - \beta(x)\gamma(x) = 1, \tag{12}$$

the action of A on $Map(\mathbb{R}, \overline{\mathbb{R}})$ is given by:

$$w(x) \overset{A}{\to} \widetilde{w}(x) = \frac{\alpha(x)w(x) + \beta(x)}{\gamma(x)w(x) + \delta(x)}. \tag{13}$$

If A acts on a solution of the Riccati equation:

$$w'(x) = a_0(x) + a_1(x)w(x) + a_2(x)w^2(x), \tag{14}$$

we obtain a solution of a new Riccati equation:

$$\widetilde{w}'(x) = \widetilde{a}_0(x) + \widetilde{a}_1(x)\widetilde{w}(x) + \widetilde{a}_2(x)\widetilde{w}^2(x), \tag{15}$$

the coefficients of which being given by

$$\overrightarrow{\widetilde{a}}(x) = M(A)\overrightarrow{a}(x) + \overrightarrow{W}(x), \quad \overrightarrow{u}(x) = \begin{pmatrix} u_2(x) \\ u_1(x) \\ u_0(x) \end{pmatrix}, \tag{16}$$

where:

$$M(A) = \begin{pmatrix} \delta^2(x) & -\gamma(x)\delta(x) & \gamma^2(x) \\ -2\beta(x)\delta(x) & \alpha(x)\delta(x) + \beta(x)\gamma(x) & -2\alpha(x)\gamma(x) \\ \beta^2(x) & -\alpha(x)\beta(x) & \alpha^2(x) \end{pmatrix}, \quad \overrightarrow{W}(x) = \begin{pmatrix} W(\gamma, \delta; x) \\ W(\delta, \alpha; x) + W(\beta, \gamma; x) \\ W(\alpha, \beta; x) \end{pmatrix} \tag{17}$$

($W(f, g; x) = f(x)g'(x) - f'(x)g(x)$ is the wronskian of $f(x)$ and $g(x)$ in x). As noted in[28], Eq.(16) defines an affine action of $\mathcal{G}$ on the set of general Riccati equations.

B. Particular case of the RS equations and finite difference Bäcklund algorithm

The most general elements of $\mathcal{G}$ preserving the subset of RS equations has been determined in[28]. Among them we find in particular the elements of the form:

$$A(\phi) = \frac{1}{\sqrt{\lambda}} \begin{pmatrix} \phi(x) & \lambda - \phi^2(x) \\ -1 & \phi(x) \end{pmatrix}, \ \lambda > 0, \tag{18}$$

where $\phi(x)$ satisfies an RS equation with the same potential as in Eq.(1) but with a shifted energy:

$$-\phi'(x) + \phi^2(x) = V(x) - (E - \lambda). \tag{19}$$

With this choice $\widetilde{w}(x)$ satisfies the RS equation:

$$-\widetilde{w}'(x) + \widetilde{w}^2(x) = \widetilde{V}_\phi(x) - \lambda, \tag{20}$$

where $\widetilde{V}_\phi(x) = V(x) + 2\phi'(x)$.

Consequently, starting from a given RS function of the discrete spectrum $w_n(x;a)$, for every value of k such that $E_k > E_n$, we can build an element $A(w_n) \in \mathcal{G}$ of the form:

$$A(w_n) = \frac{1}{\sqrt{E_k(a) - E_n(a)}} \begin{pmatrix} w_n(x;a) & E_k(a) - E_n(a) - {w_n}^2(x;a) \\ -1 & w_n(x;a) \end{pmatrix} \tag{21}$$

which transforms w_k as:

$$w_k(x;a) \overset{A(w_n)}{\to} w_k^{(n)}(x;a) = -w_n(x;a) + \frac{E_k(a) - E_n(a)}{w_n(x;a) - w_k(x;a)}, \tag{22}$$

where $w_k^{(n)}$ is a solution of the RS equation:

$$-w_k^{(n)\prime}(x;a) + \left(w_k^{(n)}(x;a)\right)^2 = V^{(n)}(x;a) - E_k(a), \tag{23}$$

with the same energy $E_l(a)$ as in Eq(1) but with a modified potential

$$V^{(n)}(x;a) = V(x;a) + 2w_n'(x;a). \tag{24}$$

This is the content of the finite-difference Bäcklund algorithm[28–33]. It transposes at the level of the RS equations the covariance of the set of Schrödinger equations under Darboux transformations[34–36]. In the following we call $A(w_n)$ a Darboux-Bäcklund Transformation (DBT).

To $V(x;a)$, $A(w_n)$ associates the (quasi)isospectral partner $V^{(n)}(x;a)$. Among the $A(w_n)$, only $A(w_0)$ leads to the regular, usual SUSY QM partner $V^{(0)}(x;a)$. The correspondence between the eigenvalues of $V(x;a)$ and $V^{(n)}(x;a)$ is direct. We also have from Eq(7) and Eq(3)

$$\psi_k^{(n)}(x;a) \sim (w_n(x;a) - w_k(x;a))\, \psi_k(x;a), \tag{25}$$

that is (see Eq(10)),

$$w_{n,k}(x;a) = w_k^{(n)}(x;a). \tag{26}$$

Then, the finite difference Bäcklund algorithm generates exactly the RS functions corresponding to the spectrum of the generalized SUSY partner $V^{(n)}$ of V.

Note that for shape invariant potentials (SIP)[1–3], $A(w_0)$ is in fact an invariance transformation of the RS equations associated to the considered family of potentials (indexed by the multiparameter a), since in this case

$$V^{(0)}(x;a) = V(x;a_1) + R(a) \tag{27}$$

and

$$w_k^{(0)}(x,a) = w_{k-1}(x,a_1), \tag{28}$$

where $a_1 = f(a)$ and $R(a)$ are two given functions of the multiparameter a.

As we noted before, starting from the RS function w_n of a regular excited bound state which has n nodes on the real domain of definition I of $V(x;a)$, we generate via $A(w_n)$ a generalized SUSY partner which presents n singularities on this domain. Nevertheless, the finite difference Bäcklund algorithm can be applied by replacing w_n by any other solution of the same RS equation Eq(1), even if this solution does not correspond to a physical state. Knowing $w_n(x,a)$, the general solution of Eq(1) is given by

$$W_n(x;a,W_0)=w_n(x;a)-\frac{e^{2\int_{x_0}^{x}w_n(s;a)ds}}{W_0+\int_{x_0}^{x}dse^{2\int_{x_0}^{s}w_n(t;a)dt}}, \tag{29}$$

where W_0 is an arbitrary real parameter. We could then use the DBT $A(W_n)$ to build a generalized SUSY partner potential $V^{(n)}(x;a,W_0)=V(x;a)+2W_n'(x;a,W_0)$ and look for values of W_0 for which W_n and $V^{(n)}$ are not singular. For some potentials it is nevertheless possible, by using specific symmetries, to build directly the researched regular RS functions. Such symmetries exist in particular for the isotonic oscillator.

IV. THE ISOTONIC OSCILLATOR

As shown in[20], the primary translationally shape invariant potentials (TSIP), for which $a_1=a+\alpha$, can be classified into two categories in which the potential can be brought into a harmonic or isotonic form respectively, using a change of variables which satisfies a constant coefficient Riccati equation.

The first element of the second category is the isotonic oscillator potential itself (ie the radial effective potential for a three dimensional isotropic harmonic oscillator with zero ground-state energy) defined on the positive real half line

$$V(x;\omega,a)=\frac{\omega^2}{4}x^2+\frac{a(a-1)}{x^2}+V_0(\omega,a),\ x>0, \tag{30}$$

with $a=l+1\geq 1$ and $V_0(\omega,a)=-\omega\left(a+\frac{1}{2}\right)$. The shape invariance property of $V(x;\omega,a)$ is expressible as

$$V^{(0)}(x;\omega,a)=V(x;\omega,a_1)+2\omega \tag{31}$$

and its spectrum is given by

$$E_n(\omega)=2n\omega,\ \psi_n(x;\omega,a)\sim\exp\left(-\int w_n(x;\omega,a)\,dx\right), \tag{32}$$

where the excited state Riccati-Schrödinger function (RS function) $w_n(x;\omega,a)$ can be written as a terminating continued fraction as

$$w_n(x;\omega,a)=w_0(x;\omega,a)+R_n(x;\omega,a), \tag{33}$$

with

$$w_0(x;\omega,a)=\frac{\omega}{2}x-\frac{a}{x} \tag{34}$$

and

$$\begin{aligned}R_n(x;\omega,a)&=-\frac{E_n(\omega)}{w_0(x;\omega,a)+w_{n-1}(x;\omega,a_1)}\\&=\frac{-2n\omega}{\omega x-(2a+1)/x-}\,\Rsh\ldots\Rsh\,\frac{2(n-j+1)\omega}{\omega x-(2(a+j)-1)/x-}\,\Rsh\ldots\Rsh\,\frac{2\omega}{\omega x-(2(a+n)-1)/x}.\end{aligned} \tag{35}$$

As it is well known, the isotonic oscillator eigenstates can be also expressed in terms of Generalized Laguerre Polynomials (GLP) $L_n^{(\lambda)}$ as

$$\psi_n(x;\omega,a) \sim x^a e^{-\omega x^2/4} L_n^{(a-1/2)}\left(\omega x^2/2\right). \tag{36}$$

This implies that we have

$$R_n(x;\omega,a) = -\left(\log\left(L_n^{(a-1/2)}\left(\omega x^2/2\right)\right)\right)' = \omega x \frac{L_{n-1}^{(a+1/2)}\left(\omega x^2/2\right)}{L_n^{(a-1/2)}\left(\omega x^2/2\right)}, \tag{37}$$

which is singular at the nodes of $\psi_n(x;\omega,a)$, that is, at the zeros of $L_n^{(a-1/2)}(\xi)$. Concerning these last, we have a classical result of Kienast,Lawton and Hahn[37–39]:

Kienast-Lawton-Hahn's Theorem

Suppose that $\alpha \notin -\mathbb{N}$. Then $L_n^{(\alpha)}(z)$ admits

1) n positive zeros if $\alpha > -1$

2) $n + [\alpha] + 1$ positive zeros if $-n < \alpha < -1$ ($[|\alpha|]$ means the integer part of α)

3) No positive zero if $\alpha < -n$

The number of negative zeros is always 0 or 1.

1) 0 if $\alpha > -1$

2) 0 if $-2k-1 < \alpha < -2k$ and 1 if $-2k < \alpha < -2k+1$, with $-n < \alpha < -1$

3) 0 if n is even and 1 if n is odd, with $\alpha < -n$

This theorem, confirms in particular that, for positive values of a, the RS function $w_n(x;\omega,a)$ corresponding to a physical bound state and then the associated generalized SUSY partner $V^{(n)}(x;\omega,a)$ of $V(x;\omega,a)$ present always n singularities on the positive half axis. This family of singular rational extensions of $V(x;\omega,a)$ will be called the $L0$ series.

V. CONFLUENT HYPERGEOMETRIC EQUATION AND ISOTONIC OSCILLATOR

The confluent hypergeometric equation

$$zy''(z;\alpha,\lambda) + (\alpha+1-z)y'(z;\alpha,\lambda) + \lambda y(z;\alpha,\lambda) = 0 \tag{38}$$

on the positive half real line, can always been reduced to a Schrödinger equation for an isotonic oscillator. Indeed, if we put $z = \omega x^2/2$ and $\phi(x;\alpha,\lambda) = y(z;\alpha,\lambda)$ in Eq.(38), we obtain the following equation for $\phi(x;\alpha,\lambda)$:

$$\phi''(x;\alpha,\lambda) + (\frac{2\alpha+1}{x} - \omega x)\phi'(x;\alpha,\lambda) - 2\omega\lambda\phi(x;\alpha,\lambda) = 0 \tag{39}$$

Then

$$\psi(x;\alpha,\lambda) \sim \phi(x;\alpha,\lambda)\exp(\frac{1}{2}\int dx(\frac{2\alpha+1}{x} - \omega x)) = x^{\alpha+1/2}e^{-\omega x^2/4}\phi(x;\alpha,\lambda) \tag{40}$$

satisfies

$$-\psi''(x;\alpha,\lambda) + \left(\frac{\omega^2 x^2}{4} + \frac{(\alpha+1/2)(\alpha-1/2)}{x^2} - \omega(\alpha+1)\right)\psi(x;\alpha,\lambda) = 2\lambda\omega\psi(x;\alpha,\lambda). \tag{41}$$

If we define $a = \alpha + 1/2$, $\psi(x;a-1/2,\lambda) = \psi_\lambda(x;a)$ and $E_\lambda(\omega) = 2\lambda\omega$, we obtain

$$H(\omega,a)\psi_\lambda(x;a) = E_\lambda(\omega)\psi_\lambda(x;a), \tag{42}$$

where

$$\psi_\lambda(x;a) \sim x^a e^{-\omega x^2/4} y(\omega x^2/2; a-1/2,\lambda), \tag{43}$$

$H(\omega,a)$ being the usual isotonic hamiltonian (see Eq.(30))

$$H(\omega,a) = -\frac{d^2}{dx^2} + V(x;\omega,a). \tag{44}$$

Eq.(42) is the Schrödinger equation for the isotonic oscillator, where for physical bound states we must have $\lambda = n$. In this case, the confluent hypergeometric equation

$$zy''(z;a-1/2,n) + (a+1/2-z)y'(z;a-1/2,n) + ny(z;a-1/2,n) = 0, \tag{45}$$

admits the regular solution

$$y(z;a-1/2,n) = L_n^{(a-1/2)}(z) \tag{46}$$

and we have

$$\psi_n(x;a) \sim x^a e^{-\omega x^2/4} L_n^{(a-1/2)}(\omega x^2/2). \tag{47}$$

This is exactly the physical state for the isotonic oscillator at the energy $E_n = 2n\omega$.

In fact, as shown by Erdelyi[4,8,38,39], Eq.(38) admits quasi rational solutions built from GLP in four sectors of the values of the parameters α and λ

$$\left\{\begin{array}{c} \lambda = n: \ y_0(z;\alpha,n) = L_n^{(\alpha)}(z) \\ \lambda = -n-\alpha-1: \ y_1(z;\alpha,\alpha+1+n) = e^z L_n^{(\alpha)}(-z) \\ \lambda = n-\alpha: \ y_2(z;\alpha,n-\alpha) = z^{-\alpha} L_n^{(-\alpha)}(z) \\ \lambda = -n-1: \ y_3(z;\alpha,-n-1) = z^{-\alpha} e^z L_n^{(-\alpha)}(-z) \end{array}\right. \tag{48}$$

They correspond to the following four eigenfunctions

$$\left\{\begin{array}{c} \psi_n(x;a) \sim x^a e^{-\omega x^2/4} L_n^{(a-1/2)}(\omega x^2/2), \quad E_n(\omega) = 2n\omega \\ \psi_{-n-\alpha-1/2}(x;a) \sim x^a e^{\omega x^2/4} L_n^{(a-1/2)}(-\omega x^2/2), \quad E_{-n-\alpha-1/2}(\omega) = -2\,(n+\alpha+1/2)\,\omega \\ \psi_{n-a+1/2}(x;a) \sim x^{1-a} e^{-\omega x^2/4} L_n^{-(a-1/2)}(\omega x^2/2), \quad E_{n-a+1/2}(\omega) = 2\,(n-a+1/2)\,\omega \\ \psi_{-n-1}(x;a) \sim x^{1-a} e^{\omega x^2/4} L_n^{-(a-1/2)}(-\omega x^2/2), \quad E_{-n-1}(\omega) = -2\,(n+1)\,\omega \end{array}\right. \tag{49}$$

The 3 last cases don't correspond to physical states and physical energies.

VI. DISCRETE SYMMETRIES OF THE ISOTONIC RS EQUATION

Since the isotonic oscillator is shape invariant, the $A(w_0)$ DBT is an invariance transformation for the RS equations associated to the family of isotonic oscillators indexed by the couple of parameters (ω,a) (see Eq.(30)). But this family of RS equations is covariant under other specific transformations which act on the parameters of the isotonic potentials and preserve their functional class. As we will see, the connections between the quasi rational sectors of the confluent hypergeometric equation admit a very simple interpretation in terms of covariance transformations of the isotonic potential.

A. Inversion of ω the parameter

The first covariance transformation for $V(x;\omega,a)$ acts on the ω parameter as

$$\omega \overset{\Gamma_\omega}{\to} (-\omega)\,, \left\{\begin{array}{c} V(x;\omega,a) \overset{\Gamma_\omega}{\to} V(x;\omega,a) + \omega(2a+1) \\ w_n(x;\omega,a) \overset{\Gamma_\omega}{\to} v_n(x;\omega,a) = w_n(x;-\omega,a), \end{array}\right. \tag{50}$$

$v_n(x;\omega,a)$ satisfying $(E_n(-\omega) = -E_n(\omega) = E_{-n}(\omega))$

$$-v_n'(x;\omega,a) + v_n^2(x;\omega,a) = V(x;\omega,a) - E_{-(n+a+1/2)}(\omega)\,. \tag{51}$$

From Eq.(33), Eq.(34) and Eq.(35), writing

$$v_n(x;\omega,a) = v_0(x;\omega,a) + Q_n(x;\omega,a), \tag{52}$$

we deduce

$$v_0(x;\omega,a) = -\frac{\omega}{2}x - \frac{a}{x} \tag{53}$$

and

$$\begin{aligned} Q_n(x;\omega,a) &= \frac{E_n(\omega)}{v_0(x;\omega,a) + v_{n-1}(x;\omega,a_1)} \\ &= -\frac{2n\omega}{\omega x + (2a+1)/x+} \Rsh \dots \Rsh \frac{2(n-j+1)\omega}{\omega x + (2(a+j)-1)/x+} \Rsh \dots \Rsh \frac{2\omega}{\omega x + (2(a+n)-1)/x} \\ &= -\left(\log\left(L_n^{(a-1/2)}\left(-\omega x^2/2\right)\right)\right)'. \end{aligned} \tag{54}$$

Clearly, for $a \geq 1$ $(l \geq 0)$, $v_n(x;\omega,a)$ does not present any singularity on the positive real half line. This result is coherent with the above mentioned Kienast-Lawton-Hahn's theorem since the argument of the GLP $L_n^{(a-1/2)}$ in the expression of Q_n is now a strictly negative value.

Note that we recover exactly the same results if we use the "spatial Wick rotation"[19,21–24]

$$w_n(x;\omega,a) \to v_n(x;\omega,a) = iw_n(ix;\omega,a). \tag{55}$$

This means that the Γ_ω transformation send the singularities of w_n, which are initially all on the real axis, on the imaginary axis. This explains why the new RS function v_n does not present any singularity on the real line. Finally, comparing Eq.(49) to Eq.(52), Eq.(53) and Eq.(54), we see that Γ_ω transforms an eigenfunction of the first sector into an eigenfunction of the second sector and then coincides with the Kummer's transformation[39].

B. Inversion of a the parameter

The second covariance transformation acts on the a parameter as

$$a \overset{\Gamma_a}{\to} 1-a, \begin{cases} V(x;\omega,a) \overset{\Gamma_a}{\to} V(x;\omega,a) + \omega(2a-1) \\ w_n(x;\omega,a) \overset{\Gamma_a}{\to} u_n(x;\omega,a) = w_n(x;\omega,1-a), \end{cases} \tag{56}$$

$u_n(x;\omega,a)$ satisfying

$$-u_n'(x;\omega,a) + u_n^2(x;\omega,a) = V(x;\omega,a) - E_{n+1/2-a}(\omega)\,. \tag{57}$$

From Eq.(33), Eq.(34) and Eq.(35) we deduce

$$u_n(x;\omega,a) = u_0(x;\omega,a) + P_n(x;\omega,a), \tag{58}$$

where

$$u_0(x;\omega,l) = \frac{\omega}{2}x + \frac{a-1}{x} \tag{59}$$

and

$$\begin{aligned} P_n(x;\omega,a) &= \frac{E_n(\omega)}{v_0(x;\omega,a)+v_{n-1}(x;\omega,a_{-1})} \\ &= \frac{-2n\omega}{\omega x+(2a-3)/x-} \Rsh \ldots \Rsh \frac{2(n-j+1)\omega}{\omega x+(2(a-j)-1)/x-} \Rsh \ldots \Rsh \frac{2\omega}{\omega x+(2(a-n)-1)/x} \\ &= -\left(\log\left(L_n^{-(a-1/2)}\left(\omega x^2/2\right)\right)\right)'. \end{aligned} \tag{60}$$

If in this case the argument of the GLP in the right hand member is strictly positive, the associated $\alpha = -(a-1/2)$ parameter being strictly negative. In accordance with the Kienast-Lawton-Hahn's theorem, by taking a sufficently large, we can decrease the number of real zeros and in particular we can eliminate all the positive zeros. Thus, if $a > n+1/2$, $L_n^{-(a-1/2)}\left(\omega x^2/2\right)$ is strictly positive for any value of x. This means that $P_n(x;\omega,a)$ and $u_n(x;\omega,a)$ are not singular on $]0,+\infty[$ when $a = n+m+1/2$, with $m>0$.

Note that

$$P_1(x;\omega,a) = \frac{-2\omega}{\omega x+(2a-3)/x} = Q_1(x;\omega,a-2). \tag{61}$$

Finally, comparing Eq.(49) to Eq.(58), Eq.(59) and Eq.(60), we see that Γ_a transforms an eigenfunction of the first sector into an eigenfunction of the third sector.

C. Inversion of both parameters ω and a

Finally, we can also act simultaneously on both parameter as

$$(\omega,a) \overset{\Gamma_a\circ\Gamma_\omega}{\rightarrow} (-\omega,1-a) \left\{ \begin{array}{c} V(x;\omega,a) \overset{\Gamma_a\circ\Gamma_\omega}{\rightarrow} V(x;\omega,a)+2\omega \\ w_n(x;\omega,a) \overset{\Gamma_a\circ\Gamma_\omega}{\rightarrow} r_n(x;\omega,a) = w_n(x;-\omega,1-a), \end{array} \right. \tag{62}$$

$r_n(x;\omega,a)$ satisfying

$$-r_n'(x;\omega,a)+r_n^2(x;\omega,a) = V(x;-\omega,1-a)-E_n(-\omega) = V(x;\omega,a)-E_{-(n+1)}(\omega). \tag{63}$$

From Eq.(33), Eq.(34) and Eq.(35) we have

$$r_n(x;\omega,a) = r_0(x;\omega,a)+T_n(x;\omega,a), \tag{64}$$

where

$$r_0(x;\omega,a) = -\frac{\omega}{2}x+\frac{a-1}{x} = -w_0(x;\omega,a-1) \tag{65}$$

and

$$\begin{aligned} T_n(x;\omega,a) &= \frac{E_n(\omega)}{r_0(x;\omega,a)+r_{n-1}(x;\omega,a_{-1})} \\ &= \frac{2n\omega}{\omega x-(2a-3)/x+} \Rsh \ldots \Rsh \frac{2(n-j+1)\omega}{\omega x-(2(a-j)-1)/x+} \Rsh \ldots \Rsh \frac{2\omega}{\omega x-(2(a-n)-1)/x} \\ &= -\left(\log\left(L_n^{-(a-1/2)}\left(-\omega x^2/2\right)\right)\right)'. \end{aligned} \tag{66}$$

In this case, the argument of the GLP in the right hand member and the associated $\alpha = -(a-1/2)$ parameter are both strictly negative. In accordance with the Kienast-Lawton-Hahn's theorem, by taking a sufficently large, we can have any zero on the negative half line if n is even ($n=2l$) and one if n is odd. Thus, if $n=2l$ and $a>2l+1/2$, $L_{2l}^{-(a-1/2)}\left(-\omega x^2/2\right)$ is strictly positive for any value of x. This means that $T_{2l}(x;\omega,a)$ and $r_{2l}(x;\omega,a)$ are not singular on $]0,+\infty[$ when $a=2l+m+1/2$, with $m>0$.

Note that

$$T_1(x;\omega,a)=\frac{-2\omega}{\omega x-(2a-3)/x}=R_1(x;\omega,a-2). \tag{67}$$

Finally, comparing Eq.(49) to Eq.(64), Eq.(65) and Eq.(66), we see that $\Gamma_a\circ\Gamma_\omega$ transforms an eigenfunction of the first sector into an eigenfunction of the fourth sector and corresponds also to a Kummer's transformation[39].

VII. REGULAR RATIONAL EXTENSIONS OF THE ISOTONIC OSCILLATOR

Since the transformations considered above are covariance transformations for the family of isotonic potentials which regularize the RS functions, we can use these regularized RS functions into the finite difference Bäcklund algorithm and generate regular isospectral partners for the isotonic potential.

A. Rational extension of the $L1$ series

w_k and v_n are associated to the same potential but with different eigenvalues (cf Eq(51))

$$\left\{\begin{array}{c}-v_n'(x;\omega,a)+v_n^2(x;\omega,a)=V(x;\omega,a)-E_{-(n+a+1/2)}(\omega)\\ -w_k'(x;\omega,a)+w_k^2(x;\omega,a)=V(x;\omega,a)-E_k(\omega),\end{array}\right. \tag{68}$$

which means that we can use v_n to build a DBT $A(v_n)$ and apply it to w_k as

$$w_k(x;\omega,a)\overset{A(v_n)}{\rightarrow}w_k^{(n)}(x;\omega,a)=-v_n(x;\omega,a)+\frac{E_k(\omega)-E_{-(n+a+1/2)}(\omega)}{v_n(x;\omega,a)-w_k(x;\omega,a)}, \tag{69}$$

where $w_k^{(n)}(x;\omega,a)$ satisfies

$$-w_k^{(n)\prime}(x;\omega,a)+\left(w_k^{(n)}(x;\omega,a)\right)^2=V^{(n)}(x;\omega,a)-E_k(\omega), \tag{70}$$

with

$$V^{(n)}(x;\omega,a)=V(x;\omega,a)+2v_n'(x;\omega,a). \tag{71}$$

For every $n\geq 0$, $V^{(n)}(x;\omega,a)$ is regular on the positive half line and isospectral to $V(x;\omega,a)$

$$V^{(n)}(x;\omega,a)\underset{iso}{\equiv}V(x;\omega,a). \tag{72}$$

Clearly, $w_-^{(n)}(x;\omega,a)=-v_n(x;\omega,a)$ is also a solution of Eq(70) associated to the eigenvalue $E_{-(n+a+1/2)}(\omega)<E_0(\omega)=0$. Nevertheless, its asymptotic behaviour is similar to the one of $w_-^{(0)}(x;\omega,a)=-\omega x/2-a/x$ and consequently

$$\psi_-^{(n)}(x;\omega,a)\sim\exp\left(-\int w_-^{(n)}(x;\omega,a)dx\right) \tag{73}$$

cannot satisfy the boundary condition associated to the physically allowed eigenstates.

All the physical eigenfunctions of $H^{(n)}(\omega,a)=-d^2/dx^2+V^{(n)}(x;\omega,a)$ are then of the form

$$\psi_k^{(n)}(x;\omega,a)=\frac{1}{\sqrt{E_k(\omega)-E_{-(n+a+1/2)}(\omega)}}A(v_n)\,\psi_k(x;\omega,a),\ \ k\geq 0 \tag{74}$$

and $H^{(n)}$ is strictly isospectral to H.

Since (cf Eq(53))

$$V(x;\omega,a)+2v_0'(x;\omega,a)=V(x;\omega,a_1), \tag{75}$$

Eq(71) and Eq(72) can still be written as

$$V^{(n)}(x;\omega,a)=V(x;\omega,a_1)+2Q_n'(x;\omega,a)\underset{iso}{\equiv}V(x;\omega,a). \tag{76}$$

For instance, we have for $n=1$

$$V^{(1)}(x;\omega,a)=V(x;\omega,a_1)+\frac{4\omega}{\omega x^2+2a+1}-\frac{8\omega\,(2a+1)}{(\omega x^2+2a+1)^2} \tag{77}$$

and we recover the first rationally-extended radial oscillator obtained by Quesne[10]. For $n=2$, we have immediately from Eq.(52), Eq.(53) and Eq.(54)

$$-v_2(x;\omega,a-1)=\frac{\omega}{2}x+\frac{a-1}{x}+\frac{4\omega x\left(\omega x^2+(2a+1)\right)}{\left(\omega x^2+(2a+1)\right)^2-2\,(2a+1)}, \tag{78}$$

which corresponds to the superpotential associated to the second rationally-extended radial oscillator of the $L1$ series obtained by Quesne[10].

More generally, we have

$$Q_n(x;\omega,a)=\left(\log\left(L_n^{(a-1/2)}\left(-\omega x^2/2\right)\right)\right)'. \tag{79}$$

In Odake-Sasaki 's approach[13,16,17], this corresponds to a prepotential of the form

$$W_n\left(x;\omega,a\right)=-\frac{\omega}{4}x^2+a\log x+\log\left(L_n^{(a-1/2)}\left(-\omega x^2/2\right)\right) \tag{80}$$

and we recover (up to a shift in $a\rightarrow a+n-2$) the result obtained in[13,16,17] and[19] for the potentials associated to the L1 exceptional orthogonal polynomials.

B. Rational extension of the $L2$ series

As in the preceding case (cf Eq(57)), we can use u_n to build a DBT $A\left(u_n\right)$

$$w_k(x;\omega,a)\overset{A(u_n)}{\rightarrow}w_k^{(n)}(x;\omega,a)=-u_n(x;\omega,a)+\frac{E_k(\omega)-E_{n+1/2-a}(\omega)}{u_n(x;\omega,a)-w_k(x;\omega,a)}, \tag{81}$$

where $w_k^{(n)}(x;\omega,a)$ satisfies

$$-w_k^{(n)\prime}(x;\omega,a)+\left(w_k^{(n)}(x;\omega,a)\right)^2=U^{(n)}(x;\omega,a)-E_k(\omega), \tag{82}$$

with

$$U^{(n)}(x;\omega,a)=V(x;\omega,a)+2u_n'(x;\omega,a). \tag{83}$$

If $a>n+1/2$, $U^{(n)}(x;\omega,a)$ is regular on the positive half line and isospectral to $V(x;\omega,a)$

$$U^{(n)}(x;\omega,a) \underset{iso}{\equiv} V(x;\omega,a). \tag{84}$$

In this case, as for the $L1$ series, we see immediately that $w_{-}^{(n)}(x;\omega,a) = -u_n(x;\omega,a)$ is another solution of Eq(82) associated to the eigenvalue $E_{n+1/2-a}(\omega)(\omega) < E_0(\omega) = 0$. But here again, the asymptotic behaviour of $w_{-}^{(n)}(x;\omega,a)$ is similar to the one of $w_{-}^{(0)}(x;\omega,a) = -\omega x/2 - (a-1)/x$ and consequently

$$\psi_{-}^{(n)}(x;\omega,a) \sim \exp\left(-\int w_{-}^{(n)}(x;\omega,a)dx\right) \tag{85}$$

cannot satisfy the boundary condition associated to the physically acceptable eigenstates.

All the physical eigenfunctions of $H^{(n)}(\omega,a) = -d^2/dx^2 + U^{(n)}(x;\omega,a)$ are then of the form

$$\psi_k^{(n)}(x;\omega,a) = \frac{1}{\sqrt{E_k(\omega) - E_{n+1/2-a}(\omega)}} A\left(u_n\right)\psi_k(x;\omega,a),\ k \geq 0 \tag{86}$$

and in the $L2$ series, $H^{(n)}$ is also strictly isospectral to H.

Since (cf Eq.(59))

$$V(x;\omega,a) + 2u_0'(x;\omega,a) = V(x;\omega,a_{-1}), \tag{87}$$

using Eq.(83) and Eq.(84), we obtain

$$U^{(n)}(x;\omega,a) = V(x;\omega,a_{-1}) + 2P_n'(x;\omega,a) \underset{iso}{\equiv} V(x;\omega,a). \tag{88}$$

Note that, since $P_1(x;\omega,a) = Q_1(x;\omega,a-2)$, the first rational extension of this family has the same functional form than the first rational extension of the preceding family.

For instance, we have for $n=2$

$$P_2(x;\omega,a) = -\frac{4\omega x\left(\omega x^2 + (2a-5)\right)}{\left(\omega x^2 + (2a-5)\right)^2 + 2(2a-5)}, \tag{89}$$

which corresponds to Quesne[10] second rational extension of the $L2$ series.

We have also, by redefining $a \to n + a$

$$V(x;\omega,a_{n-1}) \underset{iso}{\equiv} V(x;\omega,a_n) + 2P_n'(x;\omega,a_n), \tag{90}$$

where

$$\begin{aligned} P_n(x;\omega,a_n) &= -\frac{2n\omega}{\omega x + (2n+2a-3)/x -} \Rsh \ldots \Rsh \frac{2\left(n-j+1\right)\omega}{\omega x + 2\left((n+a-j)-1\right)/x -} \Rsh \ldots \Rsh \frac{2\omega}{\omega x + (2a-1)/x} \\ &= -\left(\log\left(L_n^{-(a+n-1/2)}\left(\omega x^2/2\right)\right)\right)' \end{aligned} \tag{91}$$

is regular on the positive half line for $a > 0$. In Sasaki and al[14,17] formulation, we recover the associated prepotential via

$$W_n\left(x;\omega,a\right) = -\int u_n(x;\omega,a+n)dx = -\frac{\omega}{4}x^2 - \frac{a+n-1}{x} + \log\left(L_n^{-(a+n-1/2)}\left(\omega x^2/2\right)\right) \tag{92}$$

and the family of regular rational extensions obtained is exactly the $L2$ one.

C. Rational extension of the $L3$ series

Finally, w_k and r_n being also associated to the same potential but with different eigenvalues (cf Eq(63)), here again we can use r_n to build a DBT $A(r_n)$ and apply it to w_k

$$w_k(x;\omega,a) \overset{A(r_n)}{\rightarrow} w_k^{(n)}(x;\omega,a) = -r_n(x;\omega,a) + \frac{E_k(\omega) - E_{-(n+1)}(\omega)}{r_n(x;\omega,a) - w_k(x;\omega,a)}, \tag{93}$$

where $w_k^{(n)}(x;\omega,a)$ satisfies

$$-w_k^{(n)\prime}(x;\omega,a) + \left(w_k^{(n)}(x;\omega,a)\right)^2 = W^{(n)}(x;\omega,a) - E_k(\omega), \tag{94}$$

with

$$W^{(n)}(x;\omega,a) = V(x;\omega,a) + 2r_n'(x;\omega,a). \tag{95}$$

If $n = 2l$ and $a > 2l + 1/2$, $W^{(2l)}(x;\omega,a)$ is regular on the positive half line and isospectral to $V(x;\omega,a)$

$$W^{(2l)}(x;\omega,a) \underset{iso}{\equiv} V(x;\omega,a). \tag{96}$$

As for the eigenfunctions of $H^{(2l)}(\omega,a) = -d^2/dx^2 + W^{(2l)}(x;\omega,a)$ generated from those of by the DBT Eq(93), they are given by

$$\psi_k^{(2l)}(x;\omega,a) = \frac{1}{\sqrt{E_k(\omega) - E_{-(2l+1)}(\omega)}} A(r_{2l})\, \psi_k(x;\omega,a),\ \ k \geq 0 \tag{97}$$

and constitute physically allowed eigenstates. But for the $L3$ series the isospectrality is no more strict as for the preceding series. Indeed, Eq(94) is evidently satisfied by the regular RS function

$$w_-^{(2l)}(x;\omega,a) = -r_{2l}(x;\omega,a), \tag{98}$$

the asymptotic behaviour of which being identical to the one of $w_-^{(0)}(x;\omega,a) = -r_0(x;\omega,a)$. Then

$$\psi_-^{(0)}(x;\omega,a) = \exp\left(\int dx w_-^{(0)}(x;\omega,a)\right) \sim \psi_0(x;\omega,a) \tag{99}$$

and contrarily to the preceding cases

$$\psi_-^{(2l)}(x;\omega,a) = \exp\left(\int dx w_-^{(2l)}(x;\omega,a)\right) \tag{100}$$

is a physical state associated to the eigenvalue $E_{-(n+1)}(\omega) < 0$, that is, the fundamental state of the hamiltonian $H^{(2l)}(\omega,a)$. Consequently, $H^{(2l)}$ and H are only quasi-isospectral in this series, $H^{(2l)}$ admitting a supplementary energy level lower than those of H.

Since (cf Eq.(65))

$$V(x;\omega,a) + 2r_0'(x;\omega,a) = V(x;\omega,a_{-1}) - 2\omega, \tag{101}$$

using Eq.(95) and Eq.(96), we obtain

$$W^{(n)}(x;\omega,a) = V(x;\omega,a_{-1}) - 2\omega + 2P_n'(x;\omega,a) \underset{iso}{\equiv} V(x;\omega,a). \tag{102}$$

Since $T_1(x;\omega,a) = R_1(x;\omega,a-2)$, the first rational extension of this family has the same functional form than the first rational extension of the $L0$ family.

For $n=2$, we have

$$T_2(x;\omega,a) = \frac{-4\omega x\left(\omega x^2-(2a-3)\right)}{\left(\omega x^2-(2a-3)\right)^2+2(2a-3)}, \tag{103}$$

which is regular if $a \geq 2$ ($l \geq 1$) and corresponds to Quesne[10] second rational extension of the $L3$ series.

If we redefine $a \to 2l + 1/2 + a$,

$$T_{2l}(x;\omega,2l+a+1/2) = -\log\left(L_{2l}^{-(a+2l)}\left(-\omega x^2/2\right)\right)' \tag{104}$$

and $W^{(2l)}(x;\omega,a+2l+1/2)$ are regular on the positive half line for $a > 0$.

VIII. SHAPE INVARIANCE PROPERTIES OF THE EXTENSIONS OF THE ISOTONIC OSCILLATOR

As observed initially by Quesne[9,10] on the $n=1$ and $n=2$ examples, the rational extended potentials of the $L1$ and $L2$ series inherit of the shape invariance properties of the isotonic potential. Several general proofs of this result have been recently proposed[16,17], in particular by Gomez-Ullate et al[8]. In the present approach, these shape invariance properties can be derived in a very direct and transparent manner.

A. Shape invariance of the extended potentials of the $L1$ series

The superpartner of a potential of the $L1$ series $V^{(n)}(x;\omega,a) = V(x;\omega,a) + 2v_n'(x;\omega,a)$ is defined as

$$\widetilde{V}^{(n)}(x;\omega,a) = V^{(n)}(x;\omega,a) + 2w_0^{(n)\prime}(x;\omega,a),\ n \geq 0, \tag{105}$$

$w_0^{(n)}(x;\omega,a)$ (see Eq.(69)) being the RS function associated to the ground level of $V^{(n)}$ ($E_0(\omega) = 0$).

We then have

$$\begin{aligned}\widetilde{V}^{(n)}(x;\omega,a) &= V^{(n)}(x;\omega,a) - 2v_n'(x;\omega,a) - 2\left(\frac{E_{-(n+a+1/2)}(\omega)}{v_n(x;\omega,a)-w_0(x;\omega,a)}\right)' \\ &= V(x;\omega,a) - 2\left(\frac{E_{-(n+a+1/2)}(\omega)}{v_n(x;\omega,a)-w_0(x;\omega,a)}\right)'.\end{aligned} \tag{106}$$

Using Eq(53), the shape invariance property of $V(x;\omega,a)$ in Eq.(31) can also be formulated as

$$V(x;\omega,a) + 2v_0'(x;\omega,a) = V(x;\omega,a_1). \tag{107}$$

Inserting Eq(107) in Eq(106), we obtain

$$\begin{aligned}\widetilde{V}^{(n)}(x;\omega,a) &= V(x;\omega,a_1) - 2\left(\frac{E_{-(n+a+1/2)}(\omega)}{v_n(x;\omega,a)-w_0(x;\omega,a)} + v_0(x;\omega,a)\right)' \\ &= V^{(n)}(x;\omega,a_1) - 2\left(\Delta_n^1\right)',\end{aligned} \tag{108}$$

where

$$\Delta_n^1 = \frac{E_{-(n+a+1/2)}(\omega)}{v_n(x;\omega,a)-w_0(x;\omega,a)} + v_0(x;\omega,a) + v_n(x;\omega,a_1). \tag{109}$$

As an example, consider the special case $n=1$. Using Eq(54), we can write

$$\begin{aligned}\Delta_1^1 &= -2\omega\left(a+3/2\right)\frac{1}{\frac{E_1(\omega)}{v_0(x;\omega,a)+v_0(x;\omega,a_1)}-\omega x}-\omega x-\frac{2a+1}{x}+\frac{E_1(\omega)}{v_0(x;\omega,a_1)+v_0(x;\omega,a_2)} \\ &= \left(2a+3\right)\frac{\omega x+\frac{2a+1}{x}}{\omega x^2+(2a+3)}-\omega x-\frac{2a+1}{x}-\frac{2\omega x}{\omega x^2+(2a+3)}=-\omega x.\end{aligned} \tag{110}$$

We obtain

$$\widetilde{V}^{(1)}(x;\omega,a)=V^{(1)}(x;\omega,a_1)+2\omega, \tag{111}$$

which implies that $V^{(1)}(x;\omega,a)$ has the same shape invariance properties as $V(x;\omega,a)$.

More generally, using Eq(54) and defining $z=-\omega x^2/2$ and $\alpha=a+1/2$, we obtain

$$\begin{aligned}\Delta_n^1 &= E_{-(a+n+1/2)}(\omega)\frac{1}{Q_n(x;\omega,a)-\omega x}+\left(v_0(x;\omega,a)+v_0(x;\omega,a_1)\right)+Q_n(x;\omega,a_1) \\ &= \frac{2\alpha+2}{x}\frac{L_n^{(\alpha-1)}(z)}{L_{n-1}^{(\alpha)}(z)+L_n^{(\alpha-1)}(z)}-\omega x\frac{L_{n-1}^{(\alpha+1)}(z)+L_n^{(\alpha)}(z)}{L_n^{(\alpha)}(z)}-\frac{2\alpha}{x}.\end{aligned} \tag{112}$$

But the generalized Laguerre polynomials satisfy the identity

$$L_n^{(\alpha)}(z)+L_{n-1}^{(\alpha+1)}(z)=L_n^{(\alpha+1)}(z), \tag{113}$$

which gives

$$\Delta_n^1=-\omega x\frac{(\alpha+n)\,L_n^{(\alpha-1)}(z)+zL_n^{(\alpha+1)}(z)-\alpha L_n^{(\alpha)}(z)}{zL_n^{(\alpha)}(z)}. \tag{114}$$

The other fundamental recurrence

$$(n+\alpha)\,L_{n-1}^{(\alpha)}(z)-zL_n^{(\alpha+1)}(z)-(n-z)\,L_n^{(\alpha)}(z)=0, \tag{115}$$

combined with Eq(113) gives then directly

$$\Delta_n^1=-\omega x, \tag{116}$$

that is,

$$\widetilde{V}^{(n)}(x;\omega,a)=V^{(n)}(x;\omega,a_1)+2\omega. \tag{117}$$

Consequently $V^{(n)}(x;\omega,a)$ inherits of the shape invariance properties of $V(x;\omega,a)$ for every value of n.

B. Shape invariance of the extended potentials of the $L2$ series

The superpartner of a potential $U^{(n)}(x;\omega,a)=V(x;\omega,a)+2u_n'(x;\omega,a)$ of the $L2$ series is defined as

$$\widetilde{U}^{(n)}(x;\omega,a)=U^{(n)}(x;\omega,a)+2w_0^{(n)\prime}(x;\omega,a),\ \ n\geq 0, \tag{118}$$

$w_0^{(n)}(x;\omega,a)$ (see Eq.(81)) being the RS function associated to the ground level of $U^{(n)}$. Then

$$\widetilde{U}^{(n)}(x;\omega,a)=V(x;\omega,a)-2\left(\frac{E_{n+1/2-a}(\omega)}{u_n(x;\omega,a)-w_0(x;\omega,a)}\right)'. \tag{119}$$

Using as before, the shape invariance properties of $V(x;\omega,a)$, this gives

$$\begin{aligned}\widetilde{U}^{(n)}(x;\omega,a) &= V(x;\omega,a_1)-2\left(\frac{E_{n+1/2-a}(\omega)}{u_n(x;\omega,a)-w_0(x;\omega,a)}+v_0(x;\omega,a)\right) \\ &= U^{(n)}(x;\omega,a_1)-2\left(\Delta_n^2\right)',\end{aligned} \tag{120}$$

where

$$\Delta_n^2=E_{n+1/2-a}(\omega)\frac{1}{u_n(x;\omega,a)-w_0(x;\omega,a)}+v_0(x;\omega,a)+u_n(x;\omega,a_1). \tag{121}$$

Using Eq(54) and defining $z=\omega x^2/2$ and $\alpha=1/2-a$, this becomes

$$\Delta_n^2=\frac{(2n-2a+1)\,\omega}{P_n(x;\omega,a)+\frac{2a-1}{x}}+P_n(x;\omega,a_1)=\omega x\left(\frac{L_n^{(\alpha)}(z)}{L_{n-1}^{(\alpha-1)}(z)}+\frac{(n+\alpha)L_n^{(\alpha)}(z)}{-\alpha L_n^{(\alpha)}(z)+zL_{n-1}^{(\alpha+1)}(z)}\right), \tag{122}$$

But the generalized Laguerre polynomials satisfy the identity

$$zL_{n-1}^{(\alpha+1)}(z)=(n+\alpha)L_{n-1}^{(\alpha)}(z)-nL_n^{(\alpha)}(z), \tag{123}$$

which combined to Eq(113) gives

$$\Delta_n^2=-\omega x. \tag{124}$$

Then $U^{(n)}(x;\omega,a)$ has the same shape invariance properties as $V(x;\omega,a)$ for every value of n, that is

$$\widetilde{U}^{(n)}(x;\omega,a)=U^{(n)}(x;\omega,a_1)+2\omega. \tag{125}$$

C. SUSY partners of the $L3$ series extended potentials

In this case, the superpartner of the extended potential $V^{(n)}(x;\omega,a)=V(x;\omega,a)+2r_n'(x;\omega,a)$ is defined as

$$\widetilde{W}^{(n)}(x;\omega,a)=W^{(n)}(x;\omega,a)+2\left(-r_n'(x;\omega,a)\right)=V(x;\omega,a),\ n\geq 1 \tag{126}$$

since $-r_n(x;\omega,a)$ is the RS function associated to the ground level of $W^{(n)}$.

The SUSY partner of $W^{(n)}(x;\omega,a)$ is nothing but the initial potential $V(x;\omega,a)$ itself and the DBT $A\left(v_n\right)$ is the reciprocal of a SUSY partnership.

IX. CONCLUSION AND PERSPECTIVES

In this article, a new method to generate the regular rational extensions of the isotonic oscillator associated to the $L1$ and $L2$ families of exceptional Laguerre polynomials is presented. It is based on first order Darboux-Bäcklund Transformations which are built from excited states RS functions regularized by using specific symmetries of the isotonic potential. Starting from this primary shape invariant potential and using the combination of these symmetries and DBT (as covariance transformations), we generate four towers of secondary potentials, the four series $L0$, $L1$, $L2$ and $L3$. Among them, the potentials belonging to the $L1$ and $L2$ series are regular as well as half of the potentials

of the $L3$ series, the other ones being singular on the positive half line. The secondary potentials of the $L1$ and $L2$ series inherit of the same translational shape invariance properties as the primary isotonic potential.

These new potentials being obtained, it is still possible to use the Krein-Adler theorem[40,41] and its subsequent extension obtained by Samsonov[42], to generate other secondary potentials by applications of some particular n^{th} order DBT.

A similar study can be conducted for the other second category potentials (Darboux-Pöschl-Teller or Scarf hyperbolic and trigonometric) but also for the first category potentials. These last[20] include the well known case of the one-dimensional harmonic oscillator[4,19,21–24] but also the Morse potential (the regular algebraic deformations of which having already be obtained by Gomez-Ullate et al[4]), the effective radial Kepler-Coulomb potential and the Rosen-Morse potentials. This work is in progress and will be the object of a forthcoming paper.

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